Research Note : Solar rotation inversions and the relationship between a-coefficients and mode splittings

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Abstract. From the observing campaigns of a number of helioseismic telescope networks such as the Global Oscillation Network Group (GONG) and also from the Solar Heliospheric Observatory satellite (SoHO), helioseismologists now have data on in excess of 20000 oscillation frequencies ν of the modes known collectively as the 5 minute oscillations. Depending on the order and degree of the modes these data are available either as individual frequencies of modes with known radial order n, spherical harmonic degree l, and order m, or as so-called 'a-coefficients' which are obtained by fitting polynomials to ridges of power in the (l, ν) diagram. Although there is a one-to-one relationship between these two types of data, the conversion between them is not straightforward which complicates comparison of helioseismic inference between the two types of data. Their relationship with each other and with the solar rotation rate is clarified in this paper.

1. Introduction

For observations that are well resolved in space and in time the oscillations of the Sun can be projected uniquely onto its pulsation eigenmodes, which are products of functions of radius and of spherical harmonic functions. Each mode, and therefore each measured oscillation frequency, is uniquely identified by three numbers : the radial order n, and the degree l and the azimuthal order m of the spherical harmonic. For a non-rotating star the frequency is degenerate with respect to m. Rotation removes this degeneracy and the frequencies are split multiplets. Individual mode splittings D_{nlm} $(m > 0)$ are related to the mode frequencies by :

$$D_{nlm} = \frac{\nu_{nlm} - \nu_{nl-m}}{2m} \tag{1}$$

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and to the solar rotation rate $\Omega(r, \theta)$ by :

$$2\pi D_{nlm} = \int\limits_{0}^{1} \int\limits_{-1}^{1} \mathrm{d}r\,\mathrm{d}\cos\theta\, K_{nlm}(r,\theta)\Omega(r,\theta) \tag{2}$$

where r is the fractional radius and θ the colatitude. The K_{nlm} are the mode kernels for rotation given by :

$$K_{nlm}(r,\theta) \equiv F_1^{nl}(r)G_1^{lm}(\theta) + F_2^{nl}(r)G_2^{lm}(\theta) \tag{3}$$

where

$$F_1^{nl} \equiv \rho(r)r^2 \left[\xi_{nl}^2(r) - 2L^{-1}\xi_{nl}(r)\eta_{nl}(r) + \eta_{nl}^2(r)\right]/I_{nl} \tag{4}$$

and

$$F_2^{nl} \equiv \rho(r)r^2 \left[L^{-2}\eta_{nl}^2(r)\right]/I_{nl} \tag{5}$$

Here ρ is the density as a function of radius, $L^2 \equiv l(l+1)$ and ξ_{nl}, η_{nl} are the radial and horizontal components of the displacement eigenfunction in the non-rotating star, and

$$I_{nl} = \int\limits_{0}^{1} \mathrm{d}r\, \rho r^2 (\xi_{nl}^2 + \eta_{nl}^2) \tag{6}$$

Further (using $u \equiv \cos\theta$) :

$$G_1^{lm} \equiv \frac{(l - |m|)!}{(l + |m|)!}(l + 1/2)\left[P_l^m(u)\right]^2 \tag{7}$$

$$G_2^{lm} \equiv \frac{(l - |m|)!}{(l + |m|)!}(l + 1/2)\left[(1 - u^2)\left(\frac{\mathrm{d}P_l^m(u)}{\mathrm{d}u}\right)^2 + \right.$$
$$\left. 2uP_l^m(u)\frac{\mathrm{d}P_l^m(u)}{\mathrm{d}u} + \left(\frac{m^2}{1 - u^2} - L^2\right)(P_l^m(u))^2\right] \tag{8}$$

Here the $P_l^m(u)$ are associated Legendre polynomials. This separation of K_{nlm} into F_1^{nl}, F_2^{nl} and G_1^{lm}, G_2^{lm} is not unique. Many other choices are possible but this choice is particularly useful for the purposes of this paper. The

same choice is made in Pijpers & Thompson (1996). One advantage of this choice is that :

$$G_2^{lm} = \frac{1}{2}(1-u^2)\frac{\mathrm{d}^2 G_1^{lm}}{\mathrm{d}u^2} \tag{9}$$

Note that with the definitions (7), (8) of $G_{1,2}^{lm}$ the normalization is such that :

$$\int\limits_{-1}^{1} \mathrm{d}u\, G_1^{lm}(u) = 1 = -\int\limits_{-1}^{1} \mathrm{d}u\, G_2^{lm}(u) \tag{10}$$

2. The a-coefficient fitting

The a-coefficients are constructed by fitting polynomials in m to ridges of power in the (l,ν) diagram. Details of their use can be found in eg. Schou et al. (1994, hereafter SCDT). The relationship between a-coefficients and mode frequencies is :

$$\nu_{nlm} = \sum_{s=0}^{s_{\max}} a_{nl\,s}\mathcal{P}_s^{(l)}(m) \tag{11}$$

Here the polynomials $\mathcal{P}_s^{(l)}$ have degree s and are often normalized so that $\mathcal{P}_s^{(l)}(l) = l$. As noted in SCDT, who follow the projection proposed by Ritzwoller & Lavely (1991), these properties together with the orthogonality condition :

$$\sum_{m=-l}^{l} \mathcal{P}_s^{(l)}(m)\mathcal{P}_{s'}^{(l)}(m) = 0 \quad \text{for } s \neq s' \tag{12}$$

specify the polynomials $\mathcal{P}_s^{(l)}$ completely. As noted in Appendix A of SCDT the polynomials are usually constructed in an iterative process.

In practice it can be impossible, eg. for high values of l, to identify the individual peaks of power of every mode, in which case the fitting must proceed using a slightly different scheme than that described below. However at least formally the fitting always can be described as a least squares fit of polynomials to individual frequencies. In such a least-squares fit for the a-coefficients the χ^2-measure to be minimized is :

$$\chi^2 = \sum_{m=-l}^{l}\left(\nu_{nlm} - \sum_{s=0}^{s_{\max}} a_{nl\,s}\mathcal{P}_s^{(l)}(m)\right)^2 \tag{13}$$

Differentiation of Eq. (13) with respect to $a_{nl\,s}$ yields :

$$0 = \sum_{m=-l}^{l}\left(\nu_{nlm} - \sum_{s=0}^{s_{\max}} a_{nl\,s}\mathcal{P}_s^{(l)}(m)\right)\mathcal{P}_{s'}^{(l)}(m) \tag{14}$$

or :

$$\sum_{m=-l}^{l} \nu_{nlm}\mathcal{P}_{s'}^{(l)}(m) = \sum_{m=-l}^{l}\sum_{s=0}^{s_{\max}} a_{nl\,s}\mathcal{P}_s^{(l)}(m)\mathcal{P}_{s'}^{(l)}(m)$$
$$= \sum_{s=0}^{s_{\max}} a_{nl\,s}\sum_{m=-l}^{l}\mathcal{P}_s^{(l)}(m)\mathcal{P}_{s'}^{(l)}(m) \tag{15}$$

and using the orthogonality of the $\mathcal{P}$'s then leads to :

$$\sum_{m=-l}^{l} \nu_{nlm}\mathcal{P}_s^{(l)}(m) = a_{nl\,s}\sum_{m=-l}^{l}\left(\mathcal{P}_s^{(l)}(m)\right)^2 \tag{16}$$

from which is obtained :

$$a_{nl\,s} = \frac{\sum_{m=-l}^{l}\nu_{nlm}\mathcal{P}_s^{(l)}(m)}{\sum_{m'=-l}^{l}\left(\mathcal{P}_s^{(l)}(m')\right)^2} \equiv \sum_{m=-l}^{l} c_{lm}^{(s)}\nu_{nlm} \tag{17}$$

where the $c_{lm}^{(s)}$ are defined as :

$$c_{lm}^{(s)} \equiv \frac{\mathcal{P}_s^{(l)}(m)}{\sum_{m'=-l}^{l}\left(\mathcal{P}_s^{(l)}(m')\right)^2} \tag{18}$$

In the case of inversion for the determination of the solar rotation rate only the a-coefficients for odd s are of interest since these are a measure of the flow that is symmetric between northern and southern hemispheres. Since the coefficients satisfy $c_{l\,-m}^{(2s+1)} = -c_{lm}^{(2s+1)}$ (i.e. $\mathcal{P}_{2s+1}^{(l)}(-m) = -\mathcal{P}_{2s+1}^{(l)}(m)$), it is clear that linear coefficients γ_{2s+1}^m exist such that :

$$a_{nl\,2s+1} = \sum_{m=1}^{l} \gamma_{2s+1}^m D_{nlm} \tag{19}$$

and they can be obtained from the polynomials $\mathcal{P}_s^{(l)}$ by using :

$$\gamma_{2s+1}^m = \frac{m\mathcal{P}_{2s+1}^{(l)}(m)}{\sum_{m'=1}^{l}\left(\mathcal{P}_{2s+1}^{(l)}(m')\right)^2} \tag{20}$$

3. The projection of Ω

As was demonstrated by Ritzwoller & Lavely (1991) and SCDT, this choice of orthogonal fitting polynomials $\mathcal{P}_s$ leads to a_s such that they correspond one-to-one to a projection of Ω onto polynomials W_s (i.e. Eq. (3.21) of SCDT) :

$$2\pi a_{nl\,2s+1} = \int\limits_0^1 \mathrm{d}r\, K_{nl\,s}^{(a)}(r)\Omega_s(r) \tag{21}$$

so that, if the $\Omega_s(r)$ are reconstructed from a sequence of 1-dimensional inversions using the a-coefficients, $\Omega(r, \theta)$ can be reconstructed from :

$$\Omega(r, \theta) = \sum_{s=o}^{s_{\max}} \Omega_s(r) W_s(\theta) \tag{22}$$

Since the a-coefficients are independent the associated projection functions W_s must be orthogonal to the latitudinal response functions associated with the a-coefficients. Combination of Eqs. (2), (19) and (22) yields the following orthogonality conditions :

$$\int_{-1}^{1} \mathrm{d}\cos\theta \sum_{m=1}^{l} \gamma_{2s+1}^{m} G_1^{lm}(\theta) W_{s'}(\theta) = g_{1\,s}\delta_{s\,s'}$$

$$\int_{-1}^{1} \mathrm{d}\cos\theta \sum_{m=1}^{l} \gamma_{2s+1}^{m} G_2^{lm}(\theta) W_{s'}(\theta) = g_{2\,s}\delta_{s\,s'} \tag{23}$$

so that the kernels $K_{nl\,s}^{(a)}(r)$ are given by :

$$K_{nl\,s}^{(a)}(r) \equiv F_1^{nl}(r)g_{1\,s} + F_2^{nl}(r)g_{2\,s} \tag{24}$$

If conditions (23) are not satisfied there is 'cross-talk' between a-coefficients for different s. Since the set W_s is itself complete it follows from Eq. (23) that the linear sums of the G_1 and of the G_2 must be identical apart from a constant factor :

$$\sum_{m=1}^{l} \gamma_{2s+1}^{m} G_2^{lm} \equiv G_2^{(l)\,2s+1} =$$

$$\lambda G_1^{(l)\,2s+1} \equiv \lambda \sum_{m=1}^{l} \gamma_{2s+1}^{m} G_1^{lm} \tag{25}$$

Combination of Eqs. (9) and (25) yields the equation :

$$\frac{1}{2}(1-u^2)\frac{\mathrm{d}^2 G_1^{(l)}}{\mathrm{d}u^2} = \lambda G_1^{(l)} \tag{26}$$

Integrating the terms in Eq. (25) over u and using Eq. (10) shows that :

$$\lambda = -1 \quad \text{or} \quad \sum_{m=1}^{l} \gamma_{2s+1}^{m} = 0 \tag{27}$$

The regular solution of Eq. (26) is :

$$G_1^{(l)\,t} = A_t^{(l)}(1-u^2)\frac{\mathrm{d}P_t^0(u)}{\mathrm{d}u} \tag{28}$$

for $\lambda = -\frac{1}{2}t(t+1)$. It is convenient to choose $A_t^{(l)}$ such that :

$$A_t^{(l)} = \frac{(2t+1)v_t^{(l)}}{2t(t+1)} \tag{29}$$

so that the latitudinal functions ($t = 2s+1$) are :

$$G_1^{(l)\,2s+1}(u) = \frac{1}{2}\frac{(4s+3)v_{2s+1}^{(l)}}{(2s+2)(2s+1)}(1-u^2)\frac{\mathrm{d}}{\mathrm{d}u}P_{2s+1}^0(u)$$

$$G_2^{(l)\,2s+1}(u) = -\frac{1}{2}(2s+1)(2s+2)G_1^{(l)\,2s+1} \tag{30}$$

$$= -\frac{1}{4}(4s+3)v_{2s+1}^{(l)}(1-u^2)\frac{\mathrm{d}}{\mathrm{d}u}P_{2s+1}^0(u)$$

With the definition of G_1 and G_2 through expressions (30) it is simple to show that :

$$\int_{-1}^{1} \mathrm{d}u\, G_1^{(l)\,2s+1}(u) = -\int_{-1}^{1} \mathrm{d}u\, G_2^{(l)\,2s+1}(u) = v_{2s+1}^{(l)}\delta_{s0} \tag{31}$$

By integrating all terms over u in Eq. (25), and by using Eqs. (10), (25) and (31), it follows that for all $s \neq 0$ the coefficients γ satisfy $\sum_{m=1}^{l}\gamma_{2s+1}^{m} = 0$ as demanded by (27). The $v_{2s+1}^{(l)}$ are determined by the normalization of the fitting functions $\mathcal{P}$ which is demonstrated in Sect. 4. Equations (30) can also be written as :

$$G_1^{(l)\,2s+1}(u) = -\frac{1}{2}\frac{(4s+3)v_{2s+1}^{(l)}}{(2s+2)(2s+1)}(1-u^2)^{\frac{1}{2}}P_{2s+1}^1(u)$$

$$G_2^{(l)\,2s+1}(u) = \frac{1}{4}v_{2s+1}^{(l)}(4s+3)(1-u^2)^{\frac{1}{2}}P_{2s+1}^1(u) \tag{32}$$

From the orthogonality properties of associated Legendre polynomials (cf. Gradshteyn & Ryzhik, 1994) it follows that a natural choice for the projection polynomials W_s, independent of l, for Ω is :

$$W_s(u) = -(1-u^2)^{-1/2}P_{2s+1}^1(u) \tag{33}$$

From Eq. (33) it follows that then conditions (23) are satisfied with :

$$g_{1\,s} = v_{2s+1}^{(l)}$$

$$g_{2\,s} = -\frac{1}{2}v_{2s+1}^{(l)}(2s+1)(2s+2) \tag{34}$$

4. Normalization of the kernels

The freedom of choosing the factors $v_{2s+1}^{(l)}$ introduced in Eq. (29) is directly related to the freedom of choosing the normalization of the fitting functions $\mathcal{P}_s$. Combination of expressions (25) and (32) yields :

$$\sum_{m=1}^{l} \gamma_{2s+1}^{m} G_1^{lm} = -\frac{1}{2}\frac{(4s+3)v_{2s+1}^{(l)}}{(2s+2)(2s+1)}(1-u^2)^{\frac{1}{2}}P_{2s+1}^1 \tag{35}$$

Multiplying Eq. (35) by $W_{s'}$ and integrating over u on both sides yields :

$$\sum_{m=1}^{l} \gamma_{2s+1}^{m}\beta_{2s'+1}^{lm} \equiv \sum_{m=1}^{l} \gamma_{2s+1}^{m}\int_{-1}^{1} \mathrm{d}u\, G_1^{lm}(u)W_{s'}(u)$$

$$= \delta_{ss'}v_{2s+1}^{(l)} \tag{36}$$

where the factors β_{2s+1}^{lm} are introduced as shorthand notation for the integral. Using Eq. (20) it can be seen that :

$$\sum_{m=1}^{l} m\mathcal{P}_{2s+1}^{(l)}(m)\beta_{2s+1}^{lm} = \sum_{m=1}^{l} \left(\mathcal{P}_{2s+1}^{(l)}(m)\right)^2 \tag{37}$$

From which it follows, using the orthogonality of the $\mathcal{P}$ polynomials, that :

$$\beta_{2s+1}^{lm} = \frac{1}{m}\mathcal{P}_{2s+1}^{(l)}v_{2s+1}^{(l)} \tag{38}$$

and therefore :

$$\begin{aligned}
\mathcal{P}_{2s+1}^{(l)}(m) &= m\int_{-1}^{1} du\, G_1^{lm}(u)W_s(u) \\
&= \frac{-m}{v_{2s+1}^{(l)}}\frac{(l-|m|)!}{(l+|m|)!}(l+1/2)\times \\
&\quad \int_{-1}^{1} du\,(1-u^2)^{-1/2}P_{2s+1}^1(u)\left[P_l^m(u)\right]^2
\end{aligned} \tag{39}$$

Combination of the normalization condition $\mathcal{P}_{2s+1}^{(l)}(l) = l$ with Eq. (39) yields :

$$l = -\frac{l(2l+1)}{2v_{2s+1}^{(l)}}\frac{[(2l-1)!!]^2}{2l!}\int_{-1}^{1} du\,(1-u^2)^{l-1/2}P_{2s+1}^1(u) \tag{40}$$

Equation (40) is derived by using the explicit expression for $P_l^l(u)$ (cf. Gradshteyn & Ryzhik, 1994) :

$$P_l^l(u) = (-1)^l(2l-1)!!(1-u^2)^{l/2} \tag{41}$$

The solution of the integral in Eq. (40) is (cf. Gradshteyn & Ryzhik, 1994) :

$$\begin{aligned}
\int_{-1}^{1} du\,(1-u^2)^{l-1/2}P_{2s+1}^1(u) &= \\
\frac{2\pi\Gamma(l+1)\Gamma(l)}{\Gamma(l+s+\frac{3}{2})\Gamma(l-s)\Gamma(s+1)\Gamma(-\frac{1}{2}-s)}
\end{aligned} \tag{42}$$

which means that the coefficients $v_{2s+1}^{(l)}$ must satisfy the following expression :

$$v_{2s+1}^{(l)} = (-1)^s\frac{1}{l}\frac{(2l+1)!(2s+2)!(l+s+1)!}{s!(s+1)!(l-s-1)!(2l+2s+2)!} \tag{43}$$
$$\text{for } s \leq l-1$$

For $s = 0$ Eq. (43) reduces to $v_1^{(l)} = 1$. By using that $P_1^1(u) = -(1-u^2)^{1/2}$ it is easy to show that $\mathcal{P}_1^{(l)}(m) = m$ which is identical to the result obtained by SCDT in their Appendix A.

5. Constructing the fitting functions

In Appendix A of SCDT a relatively straightforward technique is given for constructing the orthogonal fitting polynomials $\mathcal{P}$. It is also possible to obtain the fitting functions by a different iteration scheme, taking Eq. (39) as the starting point and making use of one of the functional relationships between Legendre polynomials of varying order and degree (cf. Gradshteyn & Ryzhik, 1994) :

$$P_{l-1}^{m+1} - P_{l+1}^{m+1} = (2l+1)\sqrt{1-u^2}P_l^m \tag{44}$$

When this relationship is applied to P_{2s+1}^1 in Eq. (39), this yields :

$$\begin{aligned}
\mathcal{P}_{2s+1}^{(l)}(m) &= \frac{v_{2s-1}^{(l)}}{v_{2s+1}^{(l)}}\mathcal{P}_{2s-1}^{(l)}(m) + \frac{m(4s+1)}{v_{2s+1}^{(l)}}\frac{(l-|m|)!}{(l+|m|)!}\times \\
&\quad (l+1/2)\int_{-1}^{1} du\,P_{2s}^0(u)\left[P_l^m(u)\right]^2
\end{aligned} \tag{45}$$

The integral can be evaluated by expanding the P_{2s} in terms of power series of $(1-u^2)$ and making repeated use of Eq. (44), now applied to the P_l^m. This leads to a weighted sum of a set of integrals of $[P_{l+k}^{m+|k|}]^2$ where k runs from $-s$ to $+s$. The integrals are easily evaluated from the normalization of associated Legendre polynomials. The evaluation of the weighting factors in terms of k, l, m, and s is rather cumbersome however, and there is a more convenient route to the same result. Integrals of products of three associated Legendre polynomials also occur regularly in quantum mechanics when adding angular momenta (cf. Merzbacher, 1970). It is straightforward to demonstrate that the integral in Eq. (45) can be written in terms of Clebsch-Gordan or Wigner coefficients, the construction schemes of which can be found in the literature (cf. Abramowitz & Stegun). Equation (45) then becomes :

$$\begin{aligned}
\mathcal{P}_{2s+1}^{(l)}(m) &= -\frac{s(2l+2s+1)}{(2s+1)(l-s)}\mathcal{P}_{2s-1}^{(l)}(m) + \frac{m(2l+1)}{v_{2s+1}^{(l)}}\times \\
&\quad \langle ll\,00|ll\,2s\,0\rangle\langle ll\,m\,-m|ll\,2s\,0\rangle
\end{aligned} \tag{46}$$

where the angular brackets are the usual notation for the Clebsch-Gordan coefficients.

6. Implications for SOLA inversion methods

All linear methods to perform inversions for rotation as a function of both radius and latitude reduce in algorithmic form to solving a large set of linear equations, ie. inverting a large matrix. The method of Subtractive Optimally Localized Averages (SOLA) (Pijpers & Thompson, 1994) is a possible inversion method which has proven to produce reliable results in other helioseismic problems. In the SOLA method the (symmetric) matrix to be inverted has as its elements integrals over r and u of all the cross products

of the individual kernels. For the 2-D inversion for solar rotation the number of elements in this matrix is of the order of 20000×20000 or more, which is prohibitive for the direct use of SOLA on this problem. However, a direct consequence of obtaining the explicit expressions (32) for the latitudinal kernels is that it can be demonstrated that this matrix has a sparse character. This can be used to make the direct use of SOLA quite feasible.

The matrix elements for the SOLA method are :

$$\mathcal{C}^{js\,j's'} \equiv \int\limits_0^1 \mathrm{d}r \int\limits_{-1}^1 \mathrm{d}u \; K_{nl\,s}(r,u) K_{n'l'\,s'}(r,u) \tag{47}$$

For convenience of notation the single multiplet index j is introduced, which runs over all combinations of the indices n, l. The $K_{nl\,s}$ can be written as :

$$K_{nl\,s}(r,u) = F_1^{nl}(r) G_1^{(l)\,2s+1}(u) + F_2^{nl}(r) G_2^{(l)\,2s+1}(u) \tag{48}$$

Substituting Eq. (48) into Eq. (47) for s and for s' leads to a sum of four integrals, and in each of the terms the integration over u is the following :

$$\int\limits_{-1}^1 \mathrm{d}u (4s+3)(1-u^2)^{1/2} P_{2s+1}^1 (4s'+3)(1-u^2)^{1/2} P_{2s'+1}^1 \tag{49}$$

Equation (49) is obtained by making use of the expressions (32) for the G functions. Making use of Eq. (44) this integral can be rewritten as :

$$\int\limits_{-1}^1 \mathrm{d}u \left[P_{2s}^2 - P_{2s+2}^2 \right] \left[P_{2s'}^2 - P_{2s'+2}^2 \right] =$$

$$\delta_{s-1\,s'} \left[\frac{-2}{4s+1} \frac{(2s+2)!}{(2s-2)!} \right] + $$

$$\delta_{ss'} \left[\frac{2}{4s+1} \frac{(2s+2)!}{(2s-2)!} + \frac{2}{4s+5} \frac{(2s+4)!}{(2s)!} \right] + \tag{50}$$

$$\delta_{s+1\,s'} \left[\frac{-2}{4s+5} \frac{(2s+4)!}{(2s)!} \right]$$

The right-hand side of Eq. (50) is derived by making use of the orthogonality properties and normalization of associated Legendre polynomials (cf. Abramowitz & Stegun, 1972), where the δ is the Kronecker symbol. From expression (50) it follows that the integral of the cross-product of two mode kernels is not equal to 0 only if s' is equal to $s-1, s$, or $s+1$. The matrix $\mathcal{C}^{js\,j's'}$ is therefore block tridiagonal.

Introducing the notation :

$$\mathcal{A}_{pq}^{jj'} \equiv \int\limits_0^1 \mathrm{d}r \; F_p^{nl} F_q^{n'l'} \tag{51}$$

with the indices p, q being either 1 or 2, the explicit expression for the diagonal blocks is :

$$\mathcal{C}^{js\,j's} = \sum_{p,q} \zeta_{pq}^D \mathcal{A}_{pq}^{jj'} \tag{52}$$

with the factors ζ_{pq}^D equal to :

$$\zeta_{11}^D \equiv v_{2s+1}^{(l)} v_{2s+1}^{(l')} \left[\frac{s(2s-1)}{(4s+1)(2s+2)(2s+1)} + \right.$$
$$\left. \frac{(s+2)(2s+3)}{(4s+5)(2s+2)(2s+1)} \right]$$

$$\zeta_{12}^D \equiv v_{2s+1}^{(l)} v_{2s+1}^{(l')} \left[-\frac{s(2s-1)}{(8s+2)} - \frac{(s+2)(2s+3)}{(8s+10)} \right] \tag{53}$$

$$\zeta_{21}^D \equiv \zeta_{12}^D$$

$$\zeta_{22}^D \equiv v_{2s+1}^{(l)} v_{2s+1}^{(l')} \left[\frac{s(s+1)(2s+1)(2s-1)}{(8s+2)} + \right.$$
$$\left. \frac{(s+2)(s+1)(2s+3)(2s+1)}{(8s+10)} \right]$$

The lower off-diagonal blocks are the transpose of the upper off-diagonal blocks : $\mathcal{C}^{js\,j's-1} = \mathcal{C}^{j's-1\,js}$. Therefore only expressions for the upper off-diagonal blocks are given here :

$$\mathcal{C}^{js\,j's-1} = \sum_{p,q} \zeta_{pq}^O \mathcal{A}_{pq}^{jj'} \tag{54}$$

with the factors ζ_{pq}^O equal to :

$$\zeta_{11}^O \equiv v_{2s+1}^{(l)} v_{2s-1}^{(l')} \frac{-1}{8s+2}$$

$$\zeta_{12}^O \equiv v_{2s+1}^{(l)} v_{2s-1}^{(l')} \frac{s(2s-1)}{8s+2}$$

$$\zeta_{21}^O \equiv v_{2s+1}^{(l)} v_{2s-1}^{(l')} \frac{(s+1)(2s+1)}{8s+2} \tag{55}$$

$$\zeta_{22}^O \equiv v_{2s+1}^{(l)} v_{2s-1}^{(l')} \frac{-s(s+1)(2s+1)(2s-1)}{8s+2}$$

Each block has a number of elements equal to $N_{mul} \times N_{mul}$ with N_{mul} the number of multiplets in the mode-set. The number of such blocks is equal to the number of a-coefficients available, which for currently available mode-sets is around 20. When compared with a direct 2-D SOLA inversion this procedure reduces the computer storage requirements by a factor which is roughly equal to the number of available a-coefficients (~ 20), and the CPU-time required for inversion of the matrix is reduced by a factor which is roughly equal to the square of that (~ 400).

7. Conclusions

In this paper it is shown that the process of fitting Ritzwoller-Lavely type a-coefficients to ridges in the (l, ν)-diagram leads to rotational mode kernels for the a-coefficients which can be expressed in closed form, as functions of associated Legendre polynomials. These explicit expressions for the kernels facilitate inversions for solar rotation as a function of radius and latitude, since it clearly is not necessary to apply the iterative procedure outlined in Appendix A of SCDT to the splittings mode kernels in order to obtain the a-coefficient kernels. For convenience the results that are relevant for 2-dimensional inversions for solar rotation from a-coefficients are collected below. The equivalent expression of Eq. (2) is :

$$2\pi a_{nl\,2s+1} \;=\; \int_0^1 \mathrm{d}r \int_{-1}^1 \mathrm{d}u\, K_{nl\,s}(r,u)\Omega(r,u) \tag{56}$$

where the 'mode kernels' are now given by :

$$K_{nl\,s}(r,u) \;\equiv\; F_1^{nl}(r)G_1^{(l)\,2s+1}(u) + F_2^{nl}(r)G_2^{(l)\,2s+1}(u) \tag{57}$$

with :

$$F_1^{nl} \;\equiv\; \rho(r)r^2 \left[\xi_{nl}^2(r) - \frac{2}{L}\xi_{nl}(r)\eta_{nl}(r) + \eta_{nl}^2(r)\right] / I_{nl}$$
$$F_2^{nl} \;\equiv\; \rho(r)r^2 \left[L^{-2}\eta_{nl}^2(r)\right] / I_{nl} \tag{58}$$

and :

$$G_1^{(l)\,2s+1} \;\equiv\; -\frac{1}{2}\frac{(4s+3)v_{2s+1}^{(l)}}{(2s+2)(2s+1)}(1-u^2)^{\frac{1}{2}}P_{2s+1}^1(u)$$
$$G_2^{(l)\,2s+1} \;\equiv\; \frac{1}{4}v_{2s+1}^{(l)}(4s+3)(1-u^2)^{\frac{1}{2}}P_{2s+1}^1(u) \tag{59}$$

The factors $v_{2s+1}^{(l)}$ are given by Eq. (43). The latitudinal functions satisfy a normalization slightly different from that expressed in condition (10) :

$$\int_{-1}^1 \mathrm{d}u\, G_1^{(l)\,2s+1}(u) = \delta_{s0} = -\int_{-1}^1 \mathrm{d}u\, G_2^{(l)\,2s+1}(u) \tag{60}$$

With these equations a 2-D inverse problem is defined which can be solved using methods identical to those used to invert for the solar rotation rate from individual frequency splittings.

Furthermore it is shown that SOLA inversions for 2-D rotation can be speeded up considerably because the matrix to be inverted in the SOLA method is shown to be symmetric block tridiagonal.

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